On the statistics of Gaussian light scattered by a Gaussian medium

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Letters to the Editor

On the statistics of Gaussian light scattered by a Gaussian medium

Abstract. The probability distribution of photon-counting relative to a Gaussian beam scattered by a Gaussian medium is evaluated.

As is well known (Mandel and Wolf 1965) the probability of counting n photons in a time interval $(0, T)$ is given by

$$p(n, T) = \frac{1}{n!}\int_0^\infty U^n \exp(-U)\, P(U)\, \mathrm{d}U \tag{1}$$

where $U = \alpha IT$, I being the intensity of the light falling on the photodetector of quantum efficiency α. Here the coherence time of the radiation is supposed to be much smaller than T and $P(U)$ represents the probability distribution of U.

We remember that the intensity of quasi-monochromatic light scattered by a fluctuating medium, whose typical frequencies are assumed to be much smaller than the mean frequency ω_0 of the incident light, can be written at point $\boldsymbol{R}$ as (see, for example, Mandel 1969)

$$I(t) = \beta I_0 |\Delta\chi(\boldsymbol{k}, t-R/c)|^2 \tag{2}$$

where I_0 is the intensity of the incident beam and β a constant associated with the geometry of the experiment; $\Delta\chi(\boldsymbol{k}, t-R/c)$ is the space-Fourier transform of the susceptibility fluctuations of the medium, which is supposed to be homogeneous and isotropic, evaluated at the wave number $\boldsymbol{k} = \boldsymbol{k}_0 - k_0\boldsymbol{R}/R$. Thus U results as the product of two independent statistical variables I_0 and $J_1 = |\Delta\chi(\boldsymbol{k}, t-R/c)|^2$, the first relative to the incident field and the second to the scattering medium. The expression of the probability distribution $p(n, T)$ reads then

$$p(n, T) = \frac{1}{n!}\int_0^\infty \gamma^n T^n I_0{}^n J_1{}^n \exp(-\gamma I_0 J_1 T) P_0(I_0) P_1(J_1)\, \mathrm{d}I_0\, \mathrm{d}J_1 \tag{3}$$

where $\gamma = \alpha\beta$.

The intensity distribution of the incident radiation can be assumed to be of the form (Mandel and Wolf 1965)

$$P_0(I_0) = \delta(I_0 - \langle I_0 \rangle) \tag{4}$$

or

$$P_0(I_0) = \frac{\exp(-I_0/\langle I_0 \rangle)}{\langle I_0 \rangle} \tag{5}$$

according to whether the source is a laser one or a Gaussian one. Furthermore, if the susceptibility fluctuations are produced by random motions of particles, the probability distribution of $|\Delta\chi(\mathbf{k}, t)|^2$ is given by the expression (Mandel 1969)

$$P_1(J_1) = \frac{\exp(-J_1/\langle J_1 \rangle)}{\langle J_1 \rangle}. \tag{6}$$

In the case of a laser beam impinging on a Gaussian medium one obviously obtains through equations (3), (4) and (6) the usual Bose–Einstein distributions for $p(n, T)$:

$$p(n, T) = \frac{1}{\langle n \rangle} \frac{1}{(1+1/\langle n \rangle)^{n+1}} \tag{7}$$

where $\langle n \rangle = \gamma \langle I_0 \rangle \langle J_1 \rangle T$ represents the average number of counts recorded in the time interval T.

In the case of a Gaussian beam scattered by a Gaussian medium, equations (3), (5) and (6) furnish

$$p(n, T) = \frac{n!}{\langle n \rangle^{1/2}} \exp\left(\frac{1}{2\langle n \rangle}\right) W_{-(n+1/2),0}\left(\frac{1}{\langle n \rangle}\right) \tag{8}$$

with $\langle n \rangle = \gamma \langle I_0 \rangle \langle J_1 \rangle T$, the $W_{k,m}$ being the Whittaker functions (Whittaker and Watson 1963).

The expression of the mth factorial moment associated with the distribution given in equation (8) can also be worked out, thus getting

$$\left\langle \frac{n!}{(n-m)!} \right\rangle = (m!)^2 \langle n \rangle^m \tag{9}$$

which can be compared with the corresponding expression for the Bose–Einstein distribution:

$$\left\langle \frac{n!}{(n-m)!} \right\rangle = m! \langle n \rangle^m. \tag{10}$$

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